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A Proposal for a Semester-Long Course:
Prime Numbers at the Secondary Level

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A Proposal for a Semester-Long Course:
Prime Numbers at the Secondary Level

by

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Report

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Dedication

To my Great Uncle Johnny who TOLD me to get my Master's degree; to my Uncle Ernest, who would be most proud; and to my mother, who taught me most everything fundamental.

Acknowledgements

Thank you mom, for all you did for me; Thank you dad for pushing me to do more. Thank you Natasha, for being my inspiration. I would also like to thank Dr. David Kessler M.D. FACC, and Dr. Juhana Karha M.D. FACC, who have enabled me to finish what I have started.

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Abstract

A Proposal for a Semester-Long Course: Prime Numbers at the Secondary Level

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Prime numbers play an integral part in many upper level mathematics courses, most notably in Number Theory. Can a course or section on prime numbers be introduced at the secondary (high school) level? This report outlines a possible course in a manner suitable for grade level instruction. These topics include: an extended section on the complete number system, a brief history of primes, their cardinality, and both the Fundamental Theorem of Arithmetic and Prime Number Theorem, the applications of primes, and the impact of primes within perfect numbers will all be explored. A brief discussion on questions that still remain relating to prime numbers will conclude this report.

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CHAPTER 1: Introduction

The current offered mathematics courses taught at the secondary level are no longer sufficient for all students in order to meet state of Texas requirements for graduation. Students must have four years of math to graduate under the recommended plan as well as pass all portions of the 11^{th} grade TAKS (Texas Assessment of Knowledge and Skills) test. The notion that all students have the intellectual capacity to take Precalculus, Calculus, or both, during their 11^{th} and 12^{th} grade years, respectively, is not always supported by student performance in Algebra I, Algebra II and Geometry. Classes such as Advanced Algebra and Math Models have come to fruition seemingly with no definite aim other than to prepare students for the state administered test needed to graduate.

Mathematics is a numerical language, and in order to speak this language, one must learn mathematics vocabulary. By not offering courses that broaden the mathematical vocabulary of students relative to their intellect (as is more so the case post-secondary), the growth of mathematical knowledge is stunted. The question then becomes which course(s) would meet such requirements? One possibility for a course is considered in this report. A course on the history, relevance, and application of prime numbers would be of significant value in furthering secondary students' mathematical foundation. Furthermore, such a course would provide an alternative to the current courses offered.

Number systems are discussed at a premium in secondary school; the fallacies in the teaching of number systems, (natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers and complex numbers), are two-fold: these systems are taught to be memorized as opposed to learned, and secondly, the applications of these systems are not implicated. Brown wrote on the administering of the prescribed topic: “There is an appropriate scheme which would encourage such a group to come to grips with some of these concepts in more of a deductive spirit” [**14**, p. 661]. Brown was referring to the concept of teaching thru an alternate “domain” to help incorporate the properties of number systems.

The “domain” in this report will be prime numbers; prime numbers are discussed more as an extension exercise in high school mathematics, leaving out their history and applications. The following will focus on the precise sequence and necessary topics of a proposed course about prime numbers. Barnett’s article about the inclusion of elementary number theory as a required course expressed that teaching this subject matter is “tangible” and would be “instructive”, “useful” and at least be as worthy a class as “Calculus for Engineers” [**6**, p.1002]. The origin of, vocabulary, and proofs of and about prime numbers, are discussed. First, an introduction to number sets and proofs are essential.

CHAPTER 2: Natural Numbers

It is evident (from the author's experience teaching the subject matter) that students do not have a grasp of the sets of numbers that make up the real number system. In high school mathematics, number systems appear within the "domain and range" unit in both algebra classes and during the "real number line" unit in geometry, but minimal time is devoted to the explanation of its *completeness*. The axiom of completeness, or *least upper bound property*, states that all non-empty sets of real numbers having an upper bound must have a *least upper bound*. The least upper bound is the smallest real number greater than the largest number in a given set. Thus, the introduction of all the number systems is a necessary starting point. The real numbers include: integers, whole numbers, rational numbers, irrational numbers, and natural numbers. All of these sets should be explored thoroughly; however, the number set most associated with prime numbers is the set of natural numbers, and will be a focal point of this report.

Natural numbers are part of the real number system; the natural numbers is the set $\mathbb{N} = \{1, 2, 3\ldots\}$. Natural Numbers are mainly used for two purposes: counting (*cardinal numbers*) and order (*ordinal numbers*) [**5**]. Ordinal numbers describe order, like a "top ten list", where as "a cardinal number is a numeral used to answer the question 'how many'?" [**5**, p. 226]. Natural numbers were originally defined by five postulates presented by Peano in 1889:

Postulate 1: 0 is a natural number
Postulate 2: If x is a natural number, there is another natural number denoted by x' (called the successor of x)
Postulate 3: $0 \neq x' = y'$. then $x' = y'$.
Postulate 4: If $x' = y'$, then $x = y$.

Postulate 5: (principle of induction)
If Q is a property which may or may not hold of natural numbers and if (a) 0 has the property Q, and (b) whenever a natural number x has the property Q, then x' has the property Q. [**2**, p. PE23]

Some would argue whether or not to include the number zero within the set of Natural Numbers, which contradicts Peano's first postulate. G.D. Duthie believes that it is the concept of an "initial member" and not the inclusion of the number zero that suffices, and once the initial member of a set is defined, this is sufficient for any number, as long as no number in the set comes before it [**5**]. Duthie wrote on the everyday usage of these terms and how it could seem illogical to include zero as an ordinal number:

> "for if 0 is the first ordinal, 1 is the second and 2 is the third, which sounds rather unsatisfactory. In fact, the initial member of a set of things to be numbered is in ordinary language always called the 1st, not the 0^{th}". [**5**, p**.** 224]

Duthie also critiqued the inclusion of zero in the cardinal numbers by stating that the answer to the question "how many are left" is not answered by "zero" but by more logical answers such as "there aren't any left" [**5**]. Duthie's argument agrees with the belief of many mathematicians that zero is not considered part of the Natural Number system.

Once the complete number system has been taught to a level competent of completing this course, an introduction to proofs is needed.

CHAPTER 3: A Brief History of Prime

Prime numbers are "natural numbers which are not multiples of any smaller integer except one" [**8**, p. 23]. By definition:

> Let p be prime and
> $p, l, m \in \mathbb{N}$, where $p > 1$.
> Then $p = lm$ such that either $l = 1$ or $m = 1$.

If a number is not a prime number, then it is said to be a *composite number*. Two numbers that have only the number one as a common divisor are said to be *relatively prime*, or *co-prime*. The characteristic of being a prime number is called a number's *primality*. "A *primality test* is an algorithm that can verify that given some integer n, we may conclude n is a prime number. A *primality proof* is a successful application of a primality test" [**13**, p. 18]. To determine a number's primality, it is necessary to consider the relative size of the number; for smaller numbers, (0 to 999), *trial division* is most effective. For larger numbers, several algorithms have been created to test primality; most notably *Fermat's Little Theorem*:

> If an integer p is a prime number, then for all integers a, dividing both a^p and a by p gives a result with the same remainder. [**7**]

Translated into modular form:

$$a^p \equiv a \pmod{p}.$$

Peterson stated that "a few composite numbers also pass the test, so further steps are needed to ensure that the target truly is a prime" [**9**, p. 266]. Furthermore, the likelihood that an integer that passes the test not be a prime number is very small.

Values of a^p that are composite in Fermat's primality test are called *Carmichael Numbers*. Such numbers are named after Robert Carmichael, who found the first such number, 561, under these circumstances [**12**]. Furthermore, the more times this process is repeated with a different value for a, then the probability is even smaller that it is composite.

Fermat's Little Theorem has also been referenced in formulating computer based algorithms to help compute large (more than 15 digits) prime numbers. Although there are many algorithms that generate prime numbers, it is important to note that there is no formula that generates all prime numbers and no composites.

An earlier concept that was used to determine a number prime is called The Sieve of Eratosthenes. Named after a Greek scholar from the third century B.C., "The Sieve of Eratosthenes represents the only known algorithm from antiquity that we would call a primality test" [**13**, p. 19].

The sieve is as follows: pick an interval from 2 to a given number; let 2 be the initial prime; cross out every multiple of 2 thereafter; choose the next number still available (which is 3, since it is not a multiple of 2), and then cross out its multiples. Continue this process until all numbers have been canceled out or all options have been exhausted. The remaining numbers are prime numbers. This process works for number less than one million, but the sieve begins to break down at that point [**4**].

The application of prime numbers is not limited to finding or generating the next prime number. Prime numbers play a greater role in mathematics and the answer lies within one of the basic mathematical operations: division.

CHAPTER 4: Of Prime Importance

Prime numbers are utilized in various ways in mathematical theorems and proofs; however, there is one theorem that demonstrates their necessity: *The Fundamental Theorem of Arithmetic* (FTA). The following lemmas are necessary before introducing the Fundamental Theorem of Arithmetic.

> LEMMA 1. Let a be rational and let b be the least positive integer such that ba is an integer. If c and ca are integers, then $b \mid c$.
>
> PROOF. By the division algorithm there exist integers q and r such that $c = bq + r$, $0 < r < b$. Then ra is an integer since $ra = (c - bq)a = ca - (ba)q$. Hence $r = 0$ by the definition of b.
> [**3**, p. 1116]

In the above lemma, the division algorithm is introduced and used to show that c is divisible by b. This lemma is necessary to show the following:

> LEMMA 2. Let p be prime and a an integer such that $\frac{a}{p}$ is not an integer. If b is the least positive integer such that $b\,(\frac{a}{p})$ is an integer, then $b = p$.
>
> PROOF. Since $p\,(\frac{a}{p})$ is an integer, from Lemma 1 we have $\frac{b}{p}$. Hence $b = 1$ or $b = p$. But $b \neq 1$ since $\frac{a}{p}$ is not an integer. Hence $b = p$.
> [**3, p.** 1116]

This lemma states that the least positive integer b, must be a prime number. The importance of this lemma lies in the application of prime numbers; more specifically, prime numbers have mathematical application. Now that the above mentioned lemmas have been proven true, the foundation of the FTA is established.

The FTA states every positive integer greater than one is uniquely factorable into primes, apart from the order in which the factors occur [**3**]. The FTA is one of the most applicable theorems in number theory courses. It defines the nature in which prime numbers play their most vital role. Before discussing the proof of this theorem, a brief history of the development of the proof has merit.

The concepts behind the FTA have been around for some time. Euclid wrote one of the most famous mathematical texts of all time, entitled *Element*s, circa 300 B.C. Agargun and Fletcher stated that Euclid began the FTA proof but failed to prove it: "It is significant that Propositions VII.31 and VII.30 of the Elements lead immediately to their proofs (uniqueness and prime factorization, respectively) although Euclid forbears to take these steps" [**1,** p. 53]. The first clear statement and proof of the FTA was given by Carl Freidrich Gauss (1777-1855) in his book *Disquisitiones Arithmeticae* of 1801 [**1**].

The FTA, as noted above, is proven in two parts: the existence of prime factorization and its uniqueness, respectively. The existence proof is already proven in the preceding lemmas; Gauss also assumed existence to be "clear from elementary considerations" [**1,** p. 53]. Alternatively, the existence of prime factorization can also be shown by contradiction:

> Assume that the existence property does not hold, then there is a composite number which cannot be written as the product of primes. Let n be the least such number (*Well Ordering Principle*).
> Then n is not prime, and $n = n_1 n_2$ where $1 < n_1, n_2 < n$.

> From the definition of n, both n_1 and n_2 are products of primes, and hence so is n, which is a contradiction. [**1**, p. 54]

There are also several proofs of uniqueness, but *Euclid's Lemma* remains the most popular.

> Suppose $p \mid ad$, but p does not divide a.
> Therefore *the Greatest Common Divisor (GCD)* of a and p is *1*. [**1**, p.54]

At this juncture it is imperative to introduce and explain *Bezout's Identity*, which states the following:

> Let s, p, r, d and $a \in \mathbb{Z}$ and $sp + ra = d$,
> where a and p are both divisible by d,
> the GCD.

The proof can now be completed:

> *Thus, by Bezout's identity,* $sp + ra = 1$
> Multiplying thru *by d* gives $dsp + dra = d$. [**1**]

From the last line of the preceding proof, it is obvious that p divides dsp. By rewriting the second term, dra as $r\,(ad)$, and since it is also known that p divides ad, p divides $r\,(ad)$. Therefore p also divides b.

Prime numbers are invaluable; they have distinction, order, and application. The mathematical value of prime numbers aroused further questions in the field of number theory, most notably whether or not there were infinitely many primes.

CHAPTER 5: To Infinity (& Beyond)

The infinitude of prime numbers was one of the first major questions to arise regarding this number set upon its establishment. Many mathematicians set out to prove this either true or false via several methods.

Euclid sets out to prove the infinitude of prime numbers by assuming that there are a finite number of primes:

> Assume there are a finite number of prime numbers.
> Assume P = { $p_1, p_2, p_3 .... p_k$ } for some positive integer k.
> Let Q: = ($p_1 p_2 .... p_k$) + 1
> Then gcd (Q, p_i) = 1 for i = 1, 2…k.
> $\therefore$ Q has to have a prime factor different from all existing primes.
> Reductio ad absurdum. [**11**]

Saidak offers another proof using consecutive integers:

> Let $n \in \mathbb{Z}^+ > 1$.
> Since n, $n + 1$ are consecutive integers, they must be co-prime. Hence the number $N_2 = n(n + 1)$ must have at least two different prime factors.
> Since the integers $n(n + 1)$ and $n(n + 1) + 1$ are consecutive and co-prime, the number $N_3 = (n + 1)[n(n + 1) + 1]$ must have at least three different prime factors.
> Since this can be continued indefinitely, the number of primes must be infinite. [**11**]

Another version of proving infinitely many primes, from the mid-18th century, by way of Leonhard Euler:

> Assume that $p_1, ... p_n$ is a complete list of all primes,

And consider the product $\prod_{i=1}^{N}(1-\frac{1}{p_i})^{-1} = \prod_{i=1}^{N}(1+\frac{1}{p_i}+\frac{1}{p_i^2}+...)$.
Since every integer can be written as a product of primes powers, every unit fraction $\frac{1}{n}$ occurs from multiplying the terms

$$\frac{1}{p_1^{a_1}}, \frac{1}{p_2^{a_2}}, ... \frac{1}{p_n^{a_N}}$$

Therefore, if R is any positive integer,

$$\prod_{i=1}^{N}(1-\frac{1}{p_i})^{-1} \geq \sum_{n=1}^{R}\frac{1}{n}. \qquad [\mathbf{10,}\text{ p. } 601]$$

Consequently, as $R \to \infty$, the sum on the right side tends to infinity, which contradicts the preceding statement. Therefore, Euler has shown that there is no finite list of prime numbers. Goldstein makes the following assertion relating to Euler's proof: "First, it links the Fundamental Theorem of Arithmetic with the infinitude of primes. Second, it uses analytic fact, namely the divergence of the harmonic series, to conclude the arithmetic result" [**10,** p. 601]. As a result, most 19th century number theory was developed based on this proof . Euler's proof is also of importance due to its involvement in the development of the Prime Number Theorem.

CHAPTER 6: The Development of the P.N.T.

How often do prime numbers occurring? Several mathematicians aided in the development of the *Prime Number Theorem*. "The prime number theorem allows one to predict, at least in gross terms, the way in which the primes are distributed" [**10**, p. 599]. The Prime Number Theorem states that:

For all $x \in \Re$,

$$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1$$, where $\log x$ denotes the natural log of *x*.[**10,** p. 599]

"The first published statement which came close to the prime number theorem was due to Legendre in 1798" [**9,** p. 601]. Legendre stated that $\pi(x)$ is of the form $\frac{x}{A\log x + B}$ for constants *A* and *B*. Legendre revised his conjecture in 1808 by stating

$$\pi(x) = \frac{x}{\log x + A(x)},$$

where A(x) is approximately 1.08366...". [**10**]

It is believed that what Legendre meant to state was

$$\lim_{x \to \infty} A(x) = 1.08366.$$ [**10**]

Legendre, although the first to publish a conjecture on the prime number theory, was not the first to investigate it; this distinction goes to Carl Friedrich Gauss. Gauss considered the tabulation of primes a pastime. Gauss suspected that the density with which primes occurred in the neighborhood of the integer *n* was $\frac{1}{\log n}$

so that the number of primes in the interval [a, b) should be approximately equal to

$$\int_a^b \frac{dx}{\log x}. \qquad [\mathbf{10}, \text{p. } 602]$$

Gauss never published any of his work on prime numbers; only a letter from Gauss to the astronomer Encke demonstrates his work. The letter is significant because Gauss compared his approximation to $\pi(x)$ with Legendre's formula. Gauss argues that while Legendre's formula obtains a smaller error than his, its rate of increase is much greater. Furthermore, Gauss believes that numerical evidence does not support Legendre's assertion of the limiting value of $A(x)$.

Legendre's and Gauss's assertions were the first in a series of proofs that form the Prime Number Theorem. Johann Dirichlet's memoir (1837) proved Legendre's conjecture concerning the infinitude of primes in an arithmetic progression. One of Dirichlet's creations was the *Dirichlet L-Function*:

$$L(s,\lambda) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \text{ where s} > 1. \qquad [\mathbf{10}, \text{p.}604]$$

By using the newly formed L-Function in the proof of the infinitude of primes in an arithmetic progression, Dirichlet introduced a new idea into number theory that analytic methods could be usefully applied to mathematical problems.

The next mathematician to make progress toward a proof of the prime number theorem was Tchebychev. Tchebychev wrote two memoirs, written in 1851 and 1852, respectively, which contained the following functions of a real variable x:

$$\theta(x) = \sum_{p \le x} \log p \text{ and}$$

$$\psi(x) = \sum_{p^m \leq x} \log p, \qquad [\mathbf{10}, \text{p. } 606]$$

where p runs over primes and m over positive integers.

Using $\theta(x)$ and $\psi(x)$, Tchebychev proved that the prime number theorem is equivalent to either of the following:

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1,$$

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = 1.$$

Unfortunately, Tchebychev's methods were of an elementary nature, and were not powerful enough to prove the prime number theorem.

In a memoir dated 1860, B. Riemann picked up where Dirichlet left off; Dirichlet considered the functions $L(s, \chi)$ as functions of a real variable *s,* where Riemann took the decisive step in connecting arithmetic with the theory of functions of a complex variable[**10**]. Reimann's formulas are as follows:

$$\xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad [\mathbf{10}, \text{p. } 607]$$

which is known as the *Riemann zeta function.* The zeta function converges absolutely and uniformly for *s* in a compact subset of the half-plane *Re(s)* > 1. The zeta function showed a relationship between the number of zeros of $\zeta(s)$ and the distribution of primes. The initial link between the two is demonstrated in a variation of Euler's proof of the infinitude of primes:

Suppose that there were only finitely many primes $p_{1,} p_{2 \ldots\ldots} p_n$.

Since ζ (x) = $\prod_p (1-\frac{1}{p^s})^{-1}$ (Re(s) > 1),

$\zeta(s)$ would be bounded as s tends to 1,

which contradicts $\zeta(s) = \frac{1}{s-1} + a_0 + a_1(s-1) + \ldots\ldots$ [**10**, p. 607]

Clearly, the only *singularity* is at s = 1, which implies that there are an infinite number of primes.

Riemann ultimately arrived at the following formula, known as *Riemann's explicit formula:*

$$\psi(x) = x - \sum_p \frac{x^p}{p} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2}),$$

Goldstien wrote that "The formula explicitly puts in evidence form of the prime number theorem by equating $\psi(x)$ with x plus an error term which depends on the zeros of the zeta function" [**10**, p. 609].

By denoting the error term as E(x), the prime number theorem is equal to

$$\lim_{x\to\infty} \frac{E(x)}{x} = 0,$$

Which is equivalent to

$$\lim_{x\to\infty} \frac{1}{x} \sum_p \frac{x^p}{p} = 0.$$

The final step in proving the prime number theorem was taken by Hadamard and Poussin in 1896. Together they established the existence of a zero-free region for $\zeta(s)$.

Furthermore, Hadamard and Poussin showed that

$\exists$ constants a, t_0 such that
$\zeta(\sigma + it) \neq 0$ if $\sigma \geq 1 - \frac{1}{a}\log|t|,\ |t| \geq 0$

Hadamard and Poussin were then able to conclude the proof of the prime number theorem in the following manner:

$$\psi(x) = x + O(xe^{-c(\log x)^{1/14}}).$$

Many mathematicians have applied prime numbers to some aspect of their work; it would be unfair to all the mathematicians who contributed to this conclusion to give the credit solely to Hadamard and Poussin [**10**]. "For at each step in the chain of discovery, brilliant and fertile ideas were discovered, and provided the material out of which to fashion the next link" [**10**, p. 610]. This statement is demonstrated in the following section, as mathematicians whose work using prime numbers are of enough significance that their respective conjectures are named after them.

CHAPTER 7: Famous Prime Conjectures

Goldbach's Conjecture

"Goldbach's Conjecture is one of the most well-known unsolved mysteries in mathematics" [**8**, p. 24]. In a letter to Leonhard Euler in 1742, Goldbach proposed the following to be true:

> "Every even integer greater than 2 can be written as the sum of two primes". [**8**, p. 24]

This is referred to as *Goldbach's strong conjecture*. The accepted belief amongst mathematicians seems to be that Goldbach's conjecture is indeed true. The conjecture has been proven correct for all even integers up to 20 billion. There have also been proofs which have shown that if the integer is sufficiently large, it can be expressed as a sum of two primes.

Within the same letter to Euler, Goldbach raised a second conjecture, known as *Goldbach's weak conjecture*:

> Any odd integer greater than 7 can be expressed as the sum of three odd primes. [**8**, p. 24]

Chan wrote "It was shown in 1937 that this conjecture is true for any integer bigger than $3^{3^{15}}$" ; however, to check all odd integers less than this value would take an inordinate amount of time because it is "seven million digits long!" [**8**, p. 24].

There has been a vast amount of work done on both proofs with more advancement made on Goldbach's weak conjecture. Although evidence suggests that the

Goldbach's conjectures are true, no proofs as of date have been accepted within mathematics circles.

Twin Primes Conjecture

A *twin prime* is a pair of prime numbers that have a difference of two. The twin prime conjecture states that there are infinitely many twin primes.

> Proof: By the Prime Number Theorem, the number of primes that are less than n (when n is sufficiently large) is about $\frac{n}{\log n}$, or the probability that a certain s, sufficiently large odd integer is prime about $\frac{1}{\log n}$. So the probability that two consecutive sufficiently large odd integers are prime is about
> $\frac{1}{\log n} \bullet \frac{1}{\log(n+2)} \approx \frac{1}{(\log n)^2}$.
> That is given a sufficiently large odd integer n, there are about $\frac{n}{(\log n)^2}$ pairs of twin primes.
> Observe that $\lim_{n\to\infty} \frac{n}{(\log n)^2} = +\infty$. [**8**, p. 24]

The proof is suggestive that there are infinitely many twin primes, but is insufficient to be conclusive.

Fermat Primes

A *Fermat number* is any number of the form $F_n = 2^{2n} + 1$; Pierre de Fermat conjectured that all Fermat numbers were prime. For n = 0, 1, 2, 3, and 4, the Fermat numbers are indeed prime, known as *Fermat Primes*. However, Euler was the first to prove the falsity of Fermat's conjecture that every F_n is prime by pointing out that 641is a factor of F_5 [**9**]. To this day no other Fermat primes have been found. Fermat numbers are also relative in geometry: "In 1796, Gauss conjectured that a regular polygon with a prime number *p* of sides is constructible if and only if *p* is a Fermat prime, which has been proven true by Gauss and Wantzel in 1837.

Mersenne Primes

This conjecture is from the French priest Fr. Marin Mersenne, who published this concept in the preface of *Cognita Physica-Matematica* [**9**]. In this publication, Mersenne only made five mistakes- the inclusion of M_{67} and M_{257} was erroneous and failed to include M_{61}, M_{89}, M_{107} [**9**]. **"**A *Mersenne number* is any integer of the form $M_p = 2^n - 1$, where $n \geq 1$" [**9**, p. 677]. A *Mersenne Prime* is any Mersenne number that is prime, which occurs when *p* is prime, but not for every *p*. An example of this case follows:

$$M_{11} = 2^{11} - 1 = 2047, \text{ which is not prime.} \qquad [\textbf{15}, \text{p.}166]$$

Since there is no definite rule to determine Mersenne Primes, the search for these numbers is a tedious task, and should come as no surprise that there is currently no proof that shows the infinitude of these primes. Currently only forty-seven Mersenne Primes are known, the largest known Mersenne Prime being $M_{43,112,609}$ which was found by GIMPS- the Great Internet Mersenne Prime Search [**15**, p.166]. The primary goal of GIMPS is to find more Mersenne Primes, as well as perfect numbers.

CHAPTER 8: Perfect Numbers

Perfect numbers are not prime numbers in themselves but derived from prime numbers in the following manner. A perfect number is defined as a number that is equal to the sum of all its divisors except itself. For example, the smallest perfect number, six, has factors of 1, 2, 3, and 6; by adding the factors together (with the exception of 6), totals six. Euclid proved that an even number is perfect provided that it is of the form $2^{n-1}(2^n-1)$, when 2^{n-1} is prime. The perfect numbers that have been found are all even numbers; no odd perfect numbers have been discovered, and it is strongly believed they do not exist. It has been stated that certain restrictions can be applied in order to ensure their existence, however remote the possibility.

Mollin also offers the following proof:

THEOREM. If 2^n-1 is prime, then n is prime and $2^{n-1}(2^n-1)$ is perfect.

PROOF: Since $(2^m-1) \mid (2^n-1)$ whenever $m \mid n$, then n must be prime whenever 2^n-1 is prime. (Note that, in general, if $n = \ell m$, then for any $b \ \varepsilon$ N, $b^n-1=(b^m-1)\sum_{j=1}^{\ell} b^{m(\ell-j)}$

Let S_1l be the sum of all divisors of 2^{n-1} and let S_2 be the sum of all the divisors of the prime (2^n-1). Then the sum S of all divisors of $2^{n-1}(2^n-1)$ is given by:

$$S=\sum_{\ell \mid 2^{n-1}(2^n-1)} \ell = \sum_{\ell \mid 2^{n-1} \bullet \ell' \mid (2^n-1)} \ell\ell' = \sum_{\ell \mid 2^{n-1}} \ell \sum_{\ell' \mid 2^n-1} \ell' = S_1 S_2$$

so $2^{n-1}(2^n-1)$ is perfect. [**13**, p. 21]

This proof also demonstrates the relationship between perfect numbers and Mersenne Primes. "The relationship between Mersenne primes and even perfect numbers is said to be one to one" [**15**, p.166]. This begs the question if there are infinitely many perfect numbers, which is also unproven up to now.

Prime numbers are of direct importance as a number set and of indirect importance as a source that other numerical sets can be referenced backed to; the baseline properties of such numbers can be of great benefit having studied them for a period of time longer than such is the case presently.

CHAPTER 9: A Means to An End

The only two courses offered beyond Algebra II are Precalculus and AP Calculus , with AP Statistics being a distant third. A semester long course about prime numbers would: give an alternative to the current choices, provide a boost to the foundations that secondary mathematics are predicated on, broaden the mathematical spectrum for students at a much younger age, paving the way for future mathematicians.

The mathematics curriculum at the secondary level has remained the same over the course of the last fifty years. The courses have remained the same while, arguably, the approaches have changed with the times; examples considered would be the “New Math” era of the 1960’s, and the technology implementation in the mid 1980’s. While the courses in place have withstood the tests of time, adding to these course offerings is imperative at this juncture. The United States is falling further behind from an educational standpoint in contrast to countries such as India and China, which are plowing ahead. Elective mathematics courses would provide the challenge that students need to help bridge the current gap.

In the state of Texas, TAKS testing is coming to an end within the next two years and End of Course exams will be the replacement. With this in mind, improved foundational skills will be necessary to achieve competency levels of these exams. A question to be raised is what will become of Math Models and Advanced Algebra courses whose TAKS focus are no longer necessary? Will these courses fade out making room for new courses? End of Course testing will be required in all core curriculum classes, so a curriculum for these courses must be adhered to, which is currently not the case.

Elective matheatics courses would not have end of course exams since they would be considered upper-level, and by nature, an elective. Also, such courses would have students in them who have the mental capacity to take these classes, or are generally interested in the topic, or both. Introducing new topics to students willing to learn the material would have a positive impact on the interest level of mathematics. Students may choose to be mathematics majors, or engineers, or scientists after taking such courses.

A course in prime numbers is just one of many topic classes that may be considered for future development. The order in which this report was written serves only as a means to an end, a start. Careful consideration was taken for the placement of each chapter, always considering what question comes next. There are in fact general questions that still remain unsolved:

1. To find a prime greater than a given prime
2. To find a function that yields only prime numbers
3. To find a prime which follows a given prime
4. To find the number of primes below a given limit
5. To calculate directly the prime number of a given rank [**12**, p. 143]

Any of the following questions can serve as a final exam report for a short course on prime numbers.

To include further material is adequate and may be necessary in starting such a course, yet one specific possible addition is noteworthy: the teaching of proofs of odd and even numbers. The benefits of this section would have significant bearing on the rest of the course. The placement of this proof section could be included after the prescribed section on natural numbers and should fill any time remaining in the semester. Since the only proofs seen nowadays are in Pre-AP Geometry, this would offer a good introduction to elementary proof writing and comprehension, which reiterates one of the aims of the course.

A new course or courses are highly suggested at the current time in secondary school. More specifically, a course which can improve the fundamentals of mathematics while also offering a differing avenue than was previously (or currently) the case; with this in mind, I strongly urge for consideration this proposal for a semester course on prime numbers.

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Vita

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